

Multiple brake orbits on compact convex symmetric reversible hypersurfaces in \mathbf{R}^{2n}

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Abstract

In this paper, we prove that there exist at least $\lceil \frac{n+1}{2} \rceil + 1$ geometrically distinct brake orbits on every C^2 compact convex symmetric hypersurface Σ in \mathbf{R}^{2n} for $n \geq 2$ satisfying the reversible condition $N\Sigma = \Sigma$ with $N = \text{diag}(-I_n, I_n)$. As a consequence, we show that there exist at least $\lceil \frac{n+1}{2} \rceil + 1$ geometrically distinct brake orbits in every bounded convex symmetric domain in \mathbf{R}^n with $n \geq 2$ which gives a positive answer to the Seifert conjecture of 1948 in the symmetric case for $n = 3$. As an application, for $n = 4$ and 5 , we prove that if there are exactly n geometrically distinct closed characteristics on Σ , then all of them are symmetric brake orbits after suitable time translation.

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1 Introduction

Let $V \in C^2(\mathbf{R}^n, \mathbf{R})$ and $h > 0$ such that $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$ is nonempty, bounded, open and connected. Consider the following fixed energy problem of the second order autonomous Hamiltonian system

$$\ddot{q}(t) + V'(q(t)) = 0, \quad \text{for } q(t) \in \Omega, \quad (1.1)$$

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$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h, \quad \forall t \in \mathbf{R}, \quad (1.2)$$

$$\dot{q}(0) = \dot{q}\left(\frac{\tau}{2}\right) = 0, \quad (1.3)$$

$$q\left(\frac{\tau}{2} + t\right) = q\left(\frac{\tau}{2} - t\right), \quad q(t + \tau) = q(t), \quad \forall t \in \mathbf{R}. \quad (1.4)$$

A solution (τ, q) of (1.1)-(1.4) is called a *brake orbit* in Ω . We call two brake orbits q_1 and $q_2 : \mathbf{R} \rightarrow \mathbf{R}^n$ *geometrically distinct* if $q_1(\mathbf{R}) \neq q_2(\mathbf{R})$.

We denote by $\mathcal{O}(\Omega)$ and $\tilde{\mathcal{O}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in Ω respectively.

Let $J_k = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$ and $N_k = \begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix}$ with I_k being the identity in \mathbf{R}^k . If $k = n$ we will omit the subscript k for convenience, i.e., $J_n = J$ and $N_n = N$.

The symplectic group $\text{Sp}(2k)$ for any $k \in \mathbf{N}$ is defined by

$$\text{Sp}(2n) = \{M \in \mathcal{L}(\mathbf{R}^{2k}) | M^T J_k M = J_k\},$$

where M^T is the transpose of matrix M .

For any $\tau > 0$, the symplectic path in $\text{Sp}(2k)$ starting from the identity I_{2k} is defined by

$$\mathcal{P}_\tau(2k) = \{\gamma \in C([0, \tau], \text{Sp}(2k)) | \gamma(0) = I_{2k}\}.$$

Suppose that $H \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ satisfying

$$H(Nx) = H(x), \quad \forall x \in \mathbf{R}^{2n}. \quad (1.5)$$

We consider the following fixed energy problem

$$\dot{x}(t) = JH'(x(t)), \quad (1.6)$$

$$H(x(t)) = h, \quad (1.7)$$

$$x(-t) = Nx(t), \quad (1.8)$$

$$x(\tau + t) = x(t), \quad \forall t \in \mathbf{R}. \quad (1.9)$$

A solution (τ, x) of (1.6)-(1.9) is also called a *brake orbit* on $\Sigma := \{y \in \mathbf{R}^{2n} | H(y) = h\}$.

Remark 1.1. It is well known that via

$$H(p, q) = \frac{1}{2}|p|^2 + V(q), \quad (1.10)$$

$x = (p, q)$ and $p = \dot{q}$, the elements in $\mathcal{O}(\{V < h\})$ and the solutions of (1.6)-(1.9) are one to one correspondent.

In more general setting, let Σ be a C^2 compact hypersurface in \mathbf{R}^{2n} bounding a compact set C with nonempty interior. Suppose Σ has non-vanishing Gaussian curvature and satisfies the reversible condition $N(\Sigma - x_0) = \Sigma - x_0 := \{x - x_0 | x \in \Sigma\}$ for some $x_0 \in C$. Without loss of generality, we may assume $x_0 = 0$. We denote the set of all such hypersurface in \mathbf{R}^{2n} by $\mathcal{H}_b(2n)$. For $x \in \Sigma$, let $N_\Sigma(x)$ be the unit outward normal vector at $x \in \Sigma$. Note that here by the reversible condition there holds $N_\Sigma(Nx) = NN_\Sigma(x)$. We consider the dynamics problem of finding $\tau > 0$ and an absolutely continuous curve $x : [0, \tau] \rightarrow \mathbf{R}^{2n}$ such that

$$\dot{x}(t) = JN_\Sigma(x(t)), \quad x(t) \in \Sigma, \quad (1.11)$$

$$x(-t) = Nx(t), \quad x(\tau + t) = x(t), \quad \text{for all } t \in \mathbf{R}. \quad (1.12)$$

A solution (τ, x) of the problem (1.11)-(1.12) is a special closed characteristic on Σ , here we still call it a brake orbit on Σ .

We also call two brake orbits (τ_1, x_1) and (τ_2, x_2) *geometrically distinct* if $x_1(\mathbf{R}) \neq x_2(\mathbf{R})$, otherwise we say they are equivalent. Any two equivalent brake orbits are geometrically the same. We denote by $\mathcal{J}_b(\Sigma)$ the set of all brake orbits on Σ , by $[(\tau, x)]$ the equivalent class of $(\tau, x) \in \mathcal{J}_b(\Sigma)$ in this equivalent relation and by $\tilde{\mathcal{J}}_b(\Sigma)$ the set of $[(\tau, x)]$ for all $(\tau, x) \in \mathcal{J}_b(\Sigma)$. From now on, in the notation $[(\tau, x)]$ we always assume x has minimal period τ . We also denote by $\tilde{\mathcal{J}}(\Sigma)$ the set of all geometrically distinct closed characteristics on Σ .

Let (τ, x) be a solution of (1.6)-(1.9). We consider the boundary value problem of the linearized Hamiltonian system

$$\dot{y}(t) = JH''(x(t))y(t), \quad (1.13)$$

$$y(t + \tau) = y(t), \quad y(-t) = Ny(t), \quad \forall t \in \mathbf{R}. \quad (1.14)$$

Denote by $\gamma_x(t)$ the fundamental solution of the system (1.13), i.e., $\gamma_x(t)$ is the solution of the following problem

$$\dot{\gamma}_x(t) = JH''(x(t))\gamma_x(t), \quad (1.15)$$

$$\gamma_x(0) = I_{2n}. \quad (1.16)$$

We call $\gamma_x \in C([0, \tau/2], \text{Sp}(2n))$ the *associated symplectic path* of (τ, x) .

Let $B_1^n(0)$ denote the open unit ball \mathbf{R}^n centered at the origin 0. In [20] of 1948, H. Seifert proved $\tilde{\mathcal{O}}(\Omega) \neq \emptyset$ provided $V' \neq 0$ on $\partial\Omega$, V is analytic and Ω is homeomorphic to $B_1^n(0)$. Then he proposed his famous conjecture: $\#\tilde{\mathcal{O}}(\Omega) \geq n$ under the same conditions.

After 1948, many studies have been carried out for the brake orbit problem. S. Bolotin proved first in [4](also see [5]) of 1978 the existence of brake orbits in general setting. K. Hayashi in [10], H. Gluck and W. Ziller in [8], and V. Benci in [2] in 1983-1984 proved $\#\tilde{\mathcal{O}}(\Omega) \geq 1$ if V is C^1 , $\bar{\Omega} = \{V \leq h\}$ is compact, and $V'(q) \neq 0$ for all $q \in \partial\Omega$. In 1987, P. Rabinowitz in [19] proved that if H satisfies (1.5), $\Sigma \equiv H^{-1}(h)$ is star-shaped, and $x \cdot H'(x) \neq 0$ for all $x \in \Sigma$, then $\#\tilde{\mathcal{J}}_b(\Sigma) \geq 1$. In 1987, V. Benci and F. Giannoni gave a different proof of the existence of one brake orbit in [3].

In 1989, A. Szulkin in [21] proved that $\#\tilde{\mathcal{J}}_b(H^{-1}(h)) \geq n$, if H satisfies conditions in [19] of Rabinowitz and the energy hypersurface $H^{-1}(h)$ is $\sqrt{2}$ -pinched. E. van Groesen in [9] of 1985 and A. Ambrosetti, V. Benci, Y. Long in [1] of 1993 also proved $\#\tilde{\mathcal{O}}(\Omega) \geq n$ under different pinching conditions.

Without pinching condition, in [17] Y. Long, C. Zhu and the second author of this paper proved the following result: *For $n \geq 2$, suppose H satisfies*

(H1) (smoothness) $H \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$,

(H2) (reversibility) $H(Ny) = H(y)$ for all $y \in \mathbf{R}^{2n}$.

(H3) (convexity) $H''(y)$ is positive definite for all $y \in \mathbf{R}^{2n} \setminus \{0\}$,

(H4) (symmetry) $H(-y) = H(y)$ for all $y \in \mathbf{R}^{2n}$.

Then for any given $h > \min\{H(y) \mid y \in \mathbf{R}^{2n}\}$ and $\Sigma = H^{-1}(h)$, there holds

$$\#\tilde{\mathcal{J}}_b(\Sigma) \geq 2.$$

As a consequence they also proved that: *For $n \geq 2$, suppose $V(0) = 0$, $V(q) \geq 0$, $V(-q) = V(q)$ and $V''(q)$ is positive definite for all $q \in \mathbf{R}^n \setminus \{0\}$. Then for $\Omega \equiv \{q \in \mathbf{R}^n \mid V(q) < h\}$ with $h > 0$, there holds*

$$\#\tilde{\mathcal{O}}(\Omega) \geq 2.$$

Under the same condition of [17], in 2009 Liu and Zhang in [14] proved that $\#\tilde{\mathcal{J}}_b(\Sigma) \geq \lfloor \frac{n}{2} \rfloor + 1$, also they proved $\#\tilde{\mathcal{O}}(\Omega) \geq \lfloor \frac{n}{2} \rfloor + 1$ under the same condition of [17]. Moreover if all brake orbits on Σ are nondegenerate, Liu and Zhang in [14] proved that $\#\tilde{\mathcal{J}}_b(\Sigma) \geq n + \mathfrak{A}(\Sigma)$, where $2\mathfrak{A}(\Sigma)$ is the number of geometrically distinct asymmetric brake orbits on Σ .

Definition 1.1. *We denote*

$$\mathcal{H}_b^c(2n) = \{\Sigma \in \mathcal{H}_b(2n) \mid \Sigma \text{ is strictly convex}\},$$

$$\mathcal{H}_b^{s,c}(2n) = \{\Sigma \in \mathcal{H}_b^c(2n) \mid -\Sigma = \Sigma\}.$$

Definition 1.2. For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, a brake orbit (τ, x) on Σ is called symmetric if $x(\mathbf{R}) = -x(\mathbf{R})$. Similarly, for a C^2 convex symmetric bounded domain $\Omega \subset \mathbf{R}^n$, a brake orbit $(\tau, q) \in \mathcal{O}(\Omega)$ is called symmetric if $q(\mathbf{R}) = -q(\mathbf{R})$.

Note that a brake orbit $(\tau, x) \in \mathcal{J}_b(\Sigma)$ with minimal period τ is symmetric if $x(t + \tau/2) = -x(t)$ for $t \in \mathbf{R}$, a brake orbit $(\tau, q) \in \mathcal{O}(\Omega)$ with minimal period τ is symmetric if $q(t + \tau/2) = -q(t)$ for $t \in \mathbf{R}$.

In this paper, we denote by \mathbf{N} , \mathbf{Z} , \mathbf{Q} and \mathbf{R} the sets of positive integers, integers, rational numbers and real numbers respectively. We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in \mathbf{R}^n or \mathbf{R}^{2n} , by (\cdot, \cdot) the inner product of corresponding Hilbert space. For any $a \in \mathbf{R}$, we denote $E(a) = \inf\{k \in \mathbf{Z} | k \geq a\}$ and $[a] = \sup\{k \in \mathbf{Z} | k \leq a\}$.

The following are the main results of this paper.

Theorem 1.1. For any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ with $n \geq 2$, we have

$$\#\tilde{\mathcal{J}}_b(\Sigma) \geq \left\lfloor \frac{n+1}{2} \right\rfloor + 1.$$

Corollary 1.1. Suppose $V(0) = 0$, $V(q) \geq 0$, $V(-q) = V(q)$ and $V''(q)$ is positive definite for all $q \in \mathbf{R}^n \setminus \{0\}$ with $n \geq 3$. Then for any given $h > 0$ and $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$, we have

$$\#\tilde{\mathcal{O}}(\Omega) \geq \left\lfloor \frac{n+1}{2} \right\rfloor + 1.$$

Remark 1.2. Note that for $n = 3$, Corollary 1.1 yields $\#\tilde{\mathcal{O}}(\Omega) \geq 3$, which gives a positive answer to Seifert's conjecture in the convex symmetric case.

As a consequence of Theorem 1.1, we can prove

Theorem 1.2. For $n = 4, 5$ and any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, suppose

$$\#\tilde{\mathcal{J}}(\Sigma) = n.$$

Then all of them are symmetric brake orbits after suitable translation.

Example 1.1. A typical example of $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ is the ellipsoid $\mathcal{E}_n(r)$ defined as follows. Let $r = (r_1, \dots, r_n)$ with $r_j > 0$ for $1 \leq j \leq n$. Define

$$\mathcal{E}_n(r) = \left\{ x = (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbf{R}^{2n} \mid \sum_{k=1}^n \frac{x_k^2 + y_k^2}{r_k^2} = 1 \right\}.$$

If $r_j/r_k \notin \mathbf{Q}$ whenever $j \neq k$, from [7] one can see that there are precisely n geometrically distinct symmetric brake orbits on $\mathcal{E}_n(r)$ and all of them are nondegenerate.

2 Index theories of (i_{L_j}, ν_{L_j}) and (i_ω, ν_ω)

Let $\mathcal{L}(\mathbf{R}^{2n})$ denotes the set of $2n \times 2n$ real matrices and $\mathcal{L}_s(\mathbf{R}^{2n})$ denotes its subset of symmetric ones. For any $F \in \mathcal{L}_s(\mathbf{R}^{2n})$, we denote by $m^*(F)$ the dimension of maximal positive definite subspace, negative definite subspace, and kernel of any F for $*$ = +, -, 0 respectively.

In this section, we make some preparation for the proof of Theorem 3.1 below. We first briefly review the index function (i_ω, ν_ω) and (i_{L_j}, ν_{L_j}) for $j = 0, 1$, more details can be found in [14] and [16]. Following Theorem 2.3 of [23] we study the differences $i_{L_0}(\gamma) - i_{L_1}(\gamma)$ and $i_{L_0}(\gamma) + \nu_{L_0}(\gamma) - i_{L_1}(\gamma) - \nu_{L_1}(\gamma)$ for $\gamma \in \mathcal{P}_\tau(2n)$ by compute $\text{sgn} M_\varepsilon(\gamma(\tau))$. We obtain some basic lemmas which will be used frequently in the proof of the main theorem of this paper.

For any $\omega \in \mathbf{U}$, the following codimension 1 hypersurface in $\text{Sp}(2n)$ is defined by:

$$\text{Sp}(2n)_\omega^0 = \{M \in \text{Sp}(2n) | \det(M - \omega I_{2n}) = 0\}.$$

For any two continuous path ξ and $\eta: [0, \tau] \rightarrow \text{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, their joint path is defined by

$$\eta * \xi(t) = \begin{cases} \xi(2t) & \text{if } 0 \leq t \leq \frac{\tau}{2}, \\ \eta(2t - \tau) & \text{if } \frac{\tau}{2} \leq t \leq \tau. \end{cases}$$

Given any two $(2m_k \times 2m_k)$ - matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ for $k = 1, 2$, as in [16], the \diamond -product of M_1 and M_2 is defined by the following $(2(m_1 + m_2) \times 2(m_1 + m_2))$ -matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

A special path ξ_n is defined by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n}, \quad \forall t \in [0, \tau].$$

Definition 2.1. For any $\omega \in \mathbf{U}$ and $M \in \text{Sp}(2n)$, define

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker(M - \omega I_{2n}).$$

For any $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)).$$

If $\gamma(\tau) \notin \text{Sp}(2n)_\omega^0$, we define

$$i_\omega(\gamma) = [\text{Sp}(2n)_\omega^0 : \gamma * \xi_n], \quad (2.1)$$

where the right-hand side of (2.1) is the usual homotopy intersection number and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed endpoints. If $\gamma(\tau) \in \text{Sp}(2n)_\omega^0$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{i_\omega(\beta) \mid \beta(\tau) \in U \text{ and } \beta(\tau) \notin \text{Sp}(2n)_\omega^0\}.$$

Then $(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \dots, 2n\}$, is called the index function of γ at ω .

For any $M \in \text{Sp}(2n)$ we define

$$\begin{aligned} \Omega(M) &= \{P \in \text{Sp}(2n) \mid \sigma(P) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \\ &\text{and } \nu_\lambda(P) = \nu_\lambda(M), \forall \lambda \in \sigma(M) \cap \mathbf{U}\}, \end{aligned}$$

where we denote by $\sigma(P)$ the spectrum of P .

We denote by $\Omega^0(M)$ the path connected component of $\Omega(M)$ containing M , and call it the *homotopy component* of M in $\text{Sp}(2n)$.

Definition 2.2. For any $M_1, M_2 \in \text{Sp}(2n)$, we call $M_1 \approx M_2$ if $M_1 \in \Omega^0(M_2)$.

Remark 2.1. It is easy to check that \approx is an equivalent relation. If $M_1 \approx M_2$, we have $M_1^k \approx M_2^k$ for any $k \in \mathbf{N}$ and $M_1 \diamond M_3 \approx M_2 \diamond M_4$ for $M_3 \approx M_4$. Also we have $PMP^{-1} \approx M$ for any $P, M \in \text{Sp}(2n)$.

The following symplectic matrices were introduced as *basic normal forms* in [16]:

$$\begin{aligned} D(\lambda) &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, & \lambda = \pm 2, \\ N_1(\lambda, b) &= \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, & \lambda = \pm 1, b = \pm 1, 0, \\ R(\theta) &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, & \theta \in (0, \pi) \cup (\pi, 2\pi), \\ N_2(\omega, b) &= \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, & \theta \in (0, \pi) \cup (\pi, 2\pi), \end{aligned}$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ and $b_2 \neq b_3$.

For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, *splitting number* of M at ω is defined by

$$S_M^\pm(\omega) = \lim_{\epsilon \rightarrow 0^+} i_{\omega \exp(\pm \sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma)$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

Splitting numbers possesses the following properties.

Lemma 2.1. (cf. [15], Lemma 9.1.5 and List 9.1.12 of [16]) *Splitting number $S_M^\pm(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$. For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, $S_Q^\pm(\omega) = S_M^\pm(\omega)$ if $Q \approx M$. Moreover we have*

- (1) $(S_M^+(\pm 1), S_M^-(\pm 1)) = (1, 1)$ for $M = \pm N_1(1, b)$ with $b = 1$ or 0 ;
- (2) $(S_M^+(\pm 1), S_M^-(\pm 1)) = (0, 0)$ for $M = \pm N_1(1, b)$ with $b = -1$;
- (3) $(S_M^+(e^{\sqrt{-1}\theta}), S_M^-(e^{\sqrt{-1}\theta})) = (0, 1)$ for $M = R(\theta)$ with $\theta \in (0, \pi) \cup (\pi, 2\pi)$;
- (4) $(S_M^+(\omega), S_M^-(\omega)) = (0, 0)$ for $\omega \in \mathbf{U} \setminus \mathbf{R}$ and $M = N_2(\omega, b)$ is **trivial** i.e., for sufficiently small $\alpha > 0$, $MR((t-1)\alpha)^{\diamond n}$ possesses no eigenvalues on \mathbf{U} for $t \in [0, 1]$.
- (5) $(S_M^+(\omega), S_M^-(\omega)) = (1, 1)$ for $\omega \in \mathbf{U} \setminus \mathbf{R}$ and $M = N_2(\omega, b)$ is **non-trivial**.
- (6) $(S_M^+(\omega), S_M^-(\omega)) = (0, 0)$ for any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$ with $\sigma(M) \cap \mathbf{U} = \emptyset$.
- (7) $S_{M_1 \diamond M_2}^\pm(\omega) = S_{M_1}^\pm(\omega) + S_{M_2}^\pm(\omega)$, for any $M_j \in \mathrm{Sp}(2n_j)$ with $j = 1, 2$ and $\omega \in \mathbf{U}$.

Let

$$F = \mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$$

possess the standard inner product. We define the symplectic structure of F by

$$\{v, w\} = (\mathcal{J}v, w), \quad \forall v, w \in F, \quad \text{where } \mathcal{J} = (-J) \oplus J = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$

We denote by $\mathrm{Lag}(F)$ the set of Lagrangian subspaces of F , and equip it with the topology as a subspace of the Grassmannian of all $2n$ -dimensional subspaces of F .

It is easy to check that, for any $M \in \mathrm{Sp}(2n)$ its graph

$$\mathrm{Gr}(M) \equiv \left\{ \begin{pmatrix} x \\ Mx \end{pmatrix} \mid x \in \mathbf{R}^{2n} \right\}$$

is a Lagrangian subspace of F .

Let

$$V_1 = \{0\} \times \mathbf{R}^n \times \{0\} \times \mathbf{R}^n \subset \mathbf{R}^{4n}, \quad V_2 = \mathbf{R}^n \times \{0\} \times \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{4n}.$$

By Proposition 6.1 of [18] and Lemma 2.8 and Definition 2.5 of [17], we give the following definition.

Definition 2.3. For any continuous path $\gamma \in \mathcal{P}_\tau(2n)$, we define the following Maslov-type indices:

$$\begin{aligned} i_{L_0}(\gamma) &= \mu_F^{CLM}(V_1, \text{Gr}(\gamma), [0, \tau]) - n, \\ i_{L_1}(\gamma) &= \mu_F^{CLM}(V_2, \text{Gr}(\gamma), [0, \tau]) - n, \\ \nu_{L_j}(\gamma) &= \dim(\gamma(\tau)L_j \cap L_j), \quad j = 0, 1, \end{aligned}$$

where we denote by $i_F^{CLM}(V, W, [a, b])$ the Maslov index for Lagrangian subspace path pair (V, W) in F on $[a, b]$ defined by Cappell, Lee, and Miller in [6]. For any $M \in \text{Sp}(2n)$ and $j = 0, 1$, we also denote by $\nu_{L_j}(M) = \dim(ML_j \cap L_j)$.

Definition 2.4. For two paths $\gamma_0, \gamma_1 \in \mathcal{P}_\tau(2n)$ and $j = 0, 1$, we say that they are L_j -homotopic and denoted by $\gamma_0 \sim_{L_j} \gamma_1$, if there is a continuous map $\delta : [0, 1] \rightarrow \mathcal{P}(2n)$ such that $\delta(0) = \gamma_0$ and $\delta(1) = \gamma_1$, and $\nu_{L_j}(\delta(s))$ is constant for $s \in [0, 1]$.

Lemma 2.2.([11]) (1) If $\gamma_0 \sim_{L_j} \gamma_1$, there hold

$$i_{L_j}(\gamma_0) = i_{L_j}(\gamma_1), \quad \nu_{L_j}(\gamma_0) = \nu_{L_j}(\gamma_1).$$

(2) If $\gamma = \gamma_1 \diamond \gamma_2 \in \mathcal{P}(2n)$, and correspondingly $L_j = L'_j \oplus L''_j$, then

$$i_{L_j}(\gamma) = i_{L'_j}(\gamma_1) + i_{L''_j}(\gamma_2), \quad \nu_{L_j}(\gamma) = \nu_{L'_j}(\gamma_1) + \nu_{L''_j}(\gamma_2).$$

(3) If $\gamma \in \mathcal{P}(2n)$ is the fundamental solution of

$$\dot{x}(t) = JB(t)x(t)$$

with symmetric matrix function $B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$ satisfying $b_{22}(t) > 0$ for any $t \in \mathbf{R}$, then there holds

$$i_{L_0}(\gamma) = \sum_{0 < s < 1} \nu_{L_0}(\gamma_s), \quad \gamma_s(t) = \gamma(st).$$

(4) If $b_{11}(t) > 0$ for any $t \in \mathbf{R}$, there holds

$$i_{L_1}(\gamma) = \sum_{0 < s < 1} \nu_{L_1}(\gamma_s), \quad \gamma_s(t) = \gamma(st).$$

Definition 2.5. For any $\gamma \in \mathcal{P}_\tau$ and $k \in \mathbf{N} \equiv \{1, 2, \dots\}$, in this paper the k -time iteration γ^k of $\gamma \in \mathcal{P}_\tau(2n)$ in brake orbit boundary sense is defined by $\tilde{\gamma}|_{[0, k\tau]}$ with

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t - 2j\tau)(N\gamma(\tau)^{-1}N\gamma(\tau))^j, & t \in [2j\tau, (2j+1)\tau], j = 0, 1, 2, \dots \\ N\gamma(2j\tau + 2\tau - t)N(N\gamma(\tau)^{-1}N\gamma(\tau))^{j+1}, & t \in [(2j+1)\tau, (2j+2)\tau], j = 0, 1, 2, \dots \end{cases}$$

By [17] or Corollary 5.1 of [14] $\lim_{k \rightarrow \infty} \frac{i_{L_0}(\gamma^k)}{k}$ exists, as usual we define the mean i_{L_0} index of γ by

$$\hat{i}_{L_0}(\gamma) = \lim_{k \rightarrow \infty} \frac{i_{L_0}(\gamma^k)}{k}.$$

For any $P \in \text{Sp}(2n)$ and $\varepsilon \in \mathbf{R}$, we set

$$M_\varepsilon(P) = P^T \begin{pmatrix} \sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\ -\cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} P + \begin{pmatrix} \sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\ \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix}.$$

Then we have the following

Theorem 2.1.(Theorem 2.3 of [23]) *For $\gamma \in \mathcal{P}_\tau(2k)$ with $\tau > 0$, we have*

$$i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2} \text{sgn} M_\varepsilon(\gamma(\tau)),$$

where $\text{sgn} M_\varepsilon(\gamma(\tau)) = m^+(M_\varepsilon(\gamma(\tau))) - m^-(M_\varepsilon(\gamma(\tau)))$ is the signature of the symmetric matrix $M_\varepsilon(\gamma(\tau))$ and $0 < \varepsilon \ll 1$. we also have,

$$(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma)) = \frac{1}{2} \text{sign} M_\varepsilon(\gamma(\tau)),$$

where $0 < -\varepsilon \ll 1$.

Remark 2.2. (Remark 2.1 of [23]) *For any $n_j \times n_j$ symplectic matrix P_j with $j = 1, 2$ and $n_j \in \mathbf{N}$, we have*

$$\begin{aligned} M_\varepsilon(P_1 \diamond P_2) &= M_\varepsilon(P_1) \diamond M_\varepsilon(P_2), \\ \text{sgn} M_\varepsilon(P_1 \diamond P_2) &= \text{sgn} M_\varepsilon(P_1) + \text{sgn} M_\varepsilon(P_2), \end{aligned}$$

where $\varepsilon \in \mathbf{R}$.

In the following of this section we will give some lemmas which will be used frequently in the proof of our main theorem later.

Lemma 2.3. *For $k \in \mathbf{N}$ and any symplectic matrix $P = \begin{pmatrix} I_k & 0 \\ C & I_k \end{pmatrix}$, there holds $P \approx I_2^{\diamond p} \diamond N_1(1, 1)^{\diamond q} \diamond N_1(1, -1)^{\diamond r}$ with p, q, r satisfying*

$$m^0(C) = p, \quad m^-(C) = q, \quad m^+(C) = r.$$

Proof. It is clear that

$$P \approx \begin{pmatrix} I_k & 0 \\ B & I_k \end{pmatrix},$$

where $B = \text{diag}(0, -I_{m^-(C)}, I_{m^+(C)})$. Since $J_1 N_1(1, \pm 1)(J_1)^{-1} = \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}$, by Remark 2.1 we have $N_1(1, \pm 1) \approx \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}$. Then

$$P \approx I_2^{\diamond m^0(C)} \diamond N_1(1, 1)^{\diamond m^-(C)} \diamond N_1(1, -1)^{\diamond m^+(C)}.$$

By Lemma 2.1 we have

$$S_P^+(1) = m^0(C) + m^-(C) = p + q. \quad (2.2)$$

By the definition of the relation \approx , we have

$$2p + q + r = \nu_1(P) = 2m^0(C) + m^+(C) + m^-(C). \quad (2.3)$$

Also we have

$$p + q + r = m^0(C) + m^+(C) + m^-(C) = k. \quad (2.4)$$

By (2.2)-(2.4) we have

$$m^0(C) = p, \quad m^-(C) = q, \quad m^+(C) = r.$$

The proof of Lemma 2.3 is complete. ■

Definition 2.6. We call two symplectic matrices M_1 and M_2 in $\text{Sp}(2k)$ are *special homotopic* (or (L_0, L_1) -homotopic) and denote by $M_1 \sim M_2$, if there are $P_j \in \text{Sp}(2k)$ with $P_j = \text{diag}(Q_j, (Q_j^T)^{-1})$, where Q_j is a $k \times k$ invertible real matrix, and $\det(Q_j) > 0$ for $j = 1, 2$, such that

$$M_1 = P_1 M_2 P_2.$$

It is clear that \sim is an equivalent relation.

Lemma 2.4. For $M_1, M_2 \in \text{Sp}(2k)$, if $M_1 \sim M_2$, then

$$\text{sgn} M_\varepsilon(M_1) = \text{sgn} M_\varepsilon(M_2), \quad 0 \leq |\varepsilon| \ll 1, \quad (2.5)$$

$$N_k M_1^{-1} N_k M_1 \approx N_k M_2^{-1} N_k M_2. \quad (2.6)$$

Proof. By Definition 2.6, there are $P_j \in \text{Sp}(2k)$ with $P_j = \text{diag}(Q_j, (Q_j^T)^{-1})$, Q_j being $k \times k$ invertible real matrix, and $\det(Q_j) > 0$ such that

$$M_1 = P_1 M_2 P_2.$$

Since $\det(Q_j) > 0$ for $j = 1, 2$, we can joint Q_j to I_k by invertible matrix path. Hence we can joint $P_1 M_2 P_2$ to M_2 by symplectic path preserving the nullity ν_{L_0} and ν_{L_1} . By Lemma 2.2 of [23], (2.5) holds. Since $P_j N_k = N_k P_j$ for $j = 1, 2$. Direct computation shows that

$$N_k(P_1 M_2 P_2)^{-1} N_k(P_1 M_2 P_2) = P_2^{-1} N_k M_2^{-1} N_k M_2 P_2. \quad (2.7)$$

Thus (2.6) holds from Remark 2.1. The proof of Lemma 2.4 is complete. \blacksquare

Lemma 2.5. Let $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2k)$, where A, B, C, D are all $k \times k$ matrices. Then

(i) $\frac{1}{2} \text{sgn} M_\varepsilon(P) \leq k - \nu_{L_0}(P)$, for $0 < \varepsilon \ll 1$. If $B = 0$, we have $\frac{1}{2} \text{sgn} M_\varepsilon(P) \leq 0$ for $0 < \varepsilon \ll 1$.

(ii) Let $m^+(A^T C) = q$, we have

$$\frac{1}{2} \text{sgn} M_\varepsilon(P) \leq k - q, \quad 0 \leq |\varepsilon| \ll 1. \quad (2.8)$$

Moreover if $B = 0$, we have

$$\frac{1}{2} \text{sgn} M_\varepsilon(P) \leq -q, \quad 0 < -\varepsilon \ll 1. \quad (2.9)$$

(iii) $\frac{1}{2} \text{sgn} M_\varepsilon(P) \geq \dim \ker C - k$ for $0 < \varepsilon \ll 1$, If $C = 0$, then $\frac{1}{2} \text{sgn} M_\varepsilon(P) \geq 0$ for $0 < \varepsilon \ll 1$

(iv) If both B and C are invertible, we have

$$\text{sgn} M_\varepsilon(P) = \text{sgn} M_0(P), \quad 0 \leq |\varepsilon| \ll 1.$$

Proof. Since P is symplectic, so is for P^T . From $P^T J_k P = J_k$ and $P J_k P^T = J_k$ we get $A^T C, B^T D, AB^T, CD^T$ are all symmetric matrices and

$$AD^T - BC^T = I_k, \quad A^T D - C^T B = I_k. \quad (2.10)$$

We denote by $s = \sin 2\varepsilon$ and $c = \cos 2\varepsilon$. By definition of $M_\varepsilon(P)$, we have

$$M_\varepsilon(P) = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} sI_k & -cI_k \\ -cI_k & -sI_k \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} sI_k & cI_k \\ cI_k & -sI_k \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} sI_k & -2cI_k \\ 0 & -sI_k \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} sI_k & 2cI_k \\ 0 & -sI_k \end{pmatrix} \\
&= \begin{pmatrix} sA^T A - 2cA^T C - sC^T C + sI_k & * \\ sB^T A - 2cB^T C - sD^T C & sB^T B - 2cB^T D - sD^T D - sI_k \end{pmatrix} \\
&= \begin{pmatrix} sA^T A - 2cA^T C - sC^T C + sI_k & sA^T B - 2cC^T B - sC^T D \\ sB^T A - 2cB^T C - sD^T C & sB^T B - 2cB^T D - sD^T D - sI_k \end{pmatrix}, \quad (2.11)
\end{aligned}$$

where in the second equality we have used that $P^T J_k P = J_k$, in the fourth equality we have used that $M_\varepsilon(P)$ is a symmetric matrix. So

$$M_0(P) = -2 \begin{pmatrix} A^T C & C^T B \\ B^T C & B^T D \end{pmatrix} = -2 \begin{pmatrix} C^T & 0 \\ 0 & B^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where we have used $A^T C$ is symmetric. So if both B and C are invertible, $M_0(P)$ is invertible and symmetric, its signature is invariant under small perturbation, so (iv) holds.

If $\nu_{L_0}(P) = \dim \ker B > 0$, since $B^T D = D^T B$, for any $x \in \ker B \subseteq \mathbf{R}^k$, $x \neq 0$, and $0 < \varepsilon \ll 1$, we have

$$\begin{aligned}
M_\varepsilon(P) \begin{pmatrix} 0 \\ x \end{pmatrix} \cdot \begin{pmatrix} 0 \\ x \end{pmatrix} &= (sB^T B - 2cD^T B - sD^T D - sI_k)x \cdot x \\
&= -s(D^T D + I_k)x \cdot x \\
&< 0.
\end{aligned} \quad (2.12)$$

So $M_\varepsilon(P)$ is negative definite on $(0 \oplus \ker B) \subseteq \mathbf{R}^{2k}$. Hence $m^-(M_\varepsilon(p)) \geq \dim \ker B$ which yields that $\frac{1}{2} \text{sgn} M_\varepsilon(P) \leq k - \dim \ker B = k - \nu_{L_0}(P)$, for $0 < \varepsilon \ll 1$. Thus (i) holds. Similarly we can prove (iii).

If $m^+(A^T C) = q > 0$, let $A^T C$ is positive definite on $E \subseteq \mathbf{R}^k$, then for $0 \leq |s| \ll 1$, similar to (2.12) we have $M_\varepsilon(P)$ is negative on $E \oplus 0 \subseteq \mathbf{R}^{2k}$. Hence $m^-(M_\varepsilon(P)) \geq q$, which yields (2.8).

If $B = 0$, by (2.11) we have

$$M_\varepsilon(P) = \begin{pmatrix} sA^T A - 2cA^T C - sC^T C + sI_k & -sC^T D \\ -sD^T C & -sD^T D - sI_k \end{pmatrix}. \quad (2.13)$$

Since

$$\begin{pmatrix} I_k & -C^T D(D^T D + I_k)^{-1} \\ 0 & I_k \end{pmatrix} \begin{pmatrix} sA^T A - 2cA^T C - sC^T C + sI_k & -sC^T D \\ -sD^T C & -sD^T D - sI_k \end{pmatrix}.$$

$$\begin{aligned}
& \cdot \begin{pmatrix} I_k & 0 \\ -(D^T D + I_k)^{-1} D^T C & I_k \end{pmatrix} \\
& = \begin{pmatrix} sA^T A - 2cA^T C - sC^T C + sI_k + sC^T D(D^T D + I_k)^{-1} D^T C & 0 \\ 0 & -sD^T D - sI_k \end{pmatrix} \quad (2.14)
\end{aligned}$$

for $0 < -s \ll 1$, we have

$$m^-(M_\varepsilon(P)) \geq k + m^+(A^T C) \quad (2.15)$$

which yields (2.9). So (ii) holds and the proof of Lemma 2.5 is complete. \blacksquare

Lemma 2.6. ([23]) *For $\gamma \in \mathcal{P}_\tau(2)$, $b > 0$, and $0 < \varepsilon \ll 1$ small enough we have*

$$\begin{aligned}
& \operatorname{sgn} M_{\pm\varepsilon}(R(\theta)) = 0, \quad \text{for } \theta \in \mathbf{R}, \\
& \operatorname{sgn} M_\varepsilon(P) = 0, \quad \text{if } P = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ or } \pm \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}, \\
& \operatorname{sgn} M_\varepsilon(P) = 2, \quad \text{if } P = \pm \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \\
& \operatorname{sgn} M_\varepsilon(P) = -2, \quad \text{if } P = \pm \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.
\end{aligned}$$

3 Proofs of Theorems 1.1 and 1.2.

In this section we prove Theorems 1.1 and 1.2. The proof mainly depends on the method in [14] and the following

Theorem 3.1. *For any odd number $n \geq 3$, $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, let $P = \gamma(\tau)$. If $i_{L_0} \geq 0$, $i_{L_1} \geq 0$, $i(\gamma) \geq n$, $\gamma^2(t) = \gamma(t - \tau)\gamma(\tau)$ for all $t \in [\tau, 2\tau]$, and $P \sim (-I_2) \diamond Q$ with $Q \in \operatorname{Sp}(2n - 2)$, then*

$$i_{L_1}(\gamma) + S_{P^2}^+(1) - \nu_{L_0}(\gamma) > \frac{1-n}{2}. \quad (3.1)$$

Proof. If the conclusion of Theorem 3.1 does not hold, then

$$i_{L_1}(\gamma) + S_{P^2}^+(1) - \nu_{L_0}(\gamma) \leq \frac{1-n}{2}. \quad (3.2)$$

In the following we shall obtain a contradiction from (3.2). Hence (3.1) holds and Theorem 3.1 is proved.

Since $n \geq 3$ and n is odd, in the following of the proof of Theorem 3.1 we write $n = 2p + 1$ for some $p \in \mathbf{N}$. We denote by $Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $(n-1) \times (n-1)$ matrices. Then since Q is a symplectic matrix we have

$$A^T C = C^T A, \quad B^T D = D^T B, \quad AB^T = BA^T, \quad CD^T = DC^T, \quad (3.3)$$

$$AD^T - BC^T = I_{n-1}, \quad A^T D - C^T B = I_{n-1}, \quad (3.4)$$

$$\dim \ker B = \nu_{L_0}(\gamma) - 1, \quad \dim \ker C = \nu_{L_1}(\gamma) - 1. \quad (3.5)$$

Since $\gamma^2(t) = \gamma(t - \tau)\gamma(\tau)$ for all $t \in [\tau, 2\tau]$ we have γ^2 is also the twice iteration of γ in the periodic boundary value case, so by the Bott-type formula (cf. Theorem 9.2.1 of [16]) and the proof of Lemma 4.1 of [17] we have

$$\begin{aligned} & i(\gamma^2) + 2S_{P^2}^+(1) - \nu(\gamma^2) \\ = & 2i(\gamma) + 2S_P^+(1) + \sum_{\theta \in (0, \pi)} (S_P^+(e^{\sqrt{-1}\theta}) \\ & - (\sum_{\theta \in (0, \pi)} (S_P^-(e^{\sqrt{-1}\theta}) + (\nu(P) - S_P^-(1)) + (\nu_{-1}(P) - S_P^-(-1))) \\ \geq & 2n + 2S_P^+(1) - n \\ = & n + 2S_P^+(1) \\ \geq & n, \end{aligned} \quad (3.6)$$

where we have used the condition $i(\gamma) \geq n$ and $S_{P^2}^+(1) = S_P^+(1) + S_P^+(-1)$, $\nu(\gamma^2) = \nu(\gamma) + \nu_{-1}(\gamma)$. By Proposition C of [17] and Proposition 6.1 of [14] we have

$$i_{L_0}(\gamma) + i_{L_1}(\gamma) = i(\gamma^2) - n, \quad \nu_{L_0}(\gamma) + \nu_{L_1}(\gamma) = \nu(\gamma^2). \quad (3.7)$$

So by (3.6) and (3.7) we have

$$\begin{aligned} & (i_{L_1}(\gamma) + S_{P^2}^+(1) - \nu_{L_0}(\gamma)) + (i_{L_0}(\gamma) + S_{P^2}^+(1) - \nu_{L_1}(\gamma)) \\ = & i(\gamma^2) + 2S_{P^2}^+(1) - \nu(\gamma^2) - n \\ \geq & n - n \\ = & 0. \end{aligned} \quad (3.8)$$

By Theorem 2.1 and Lemma 2.6 we have

$$(i_{L_1}(\gamma) + S_{P^2}^+(1) - \nu_{L_0}(\gamma)) - (i_{L_0}(\gamma) + S_{P^2}^+(1) - \nu_{L_1}(\gamma))$$

$$\begin{aligned}
&= i_{L_1}(\gamma) - i_{L_0}(\gamma) - \nu_{L_0}(\gamma) + \nu_{L_1}(\gamma) \\
&= -\frac{1}{2}\text{sgn}M_\varepsilon(Q) - \frac{1}{2}\text{sgn}M_\varepsilon(-I_2) \\
&= -\frac{1}{2}\text{sgn}M_\varepsilon(Q) \\
&\geq 1 - n.
\end{aligned} \tag{3.9}$$

So by (3.8) and (3.9) we have

$$i_{L_1}(\gamma) + S_{P^2}^+(1) - \nu_{L_0}(\gamma) \geq \frac{1-n}{2}. \tag{3.10}$$

By (3.2), the inequality of (3.10) must be equality. Then both (3.6) and (3.9) are equality. So we have

$$i(\gamma^2) + 2S_{P^2}^+(1) - \nu(\gamma^2) = n. \tag{3.11}$$

$$i_{L_1}(\gamma) + S_{P^2}^+(1) - \nu_{L_0}(\gamma) = \frac{1-n}{2}. \tag{3.12}$$

$$i_{L_0}(\gamma) + \nu_{L_0}(\gamma) - i_{L_1}(\gamma) - \nu_{L_1}(\gamma) = n - 1. \tag{3.13}$$

Thus by (3.6), (3.11), Theorem 1.8.10 of [16], and Lemma 2.1 we have

$$P \approx (-I_2)^{\diamond p_1} \diamond N_1(1, -1)^{\diamond p_2} \diamond N_1(-1, 1)^{\diamond p_3} \diamond R(\theta_1) \diamond R(\theta_2) \diamond \cdots \diamond R(\theta_{p_4}),$$

where $p_j \geq 0$ for $j = 1, 2, 3, 4$, $p_1 + p_2 + p_3 + p_4 = n$ and $\theta_j \in (0, \pi)$ for $1 \leq j \leq p_4$. Otherwise by (3.6) and Lemma 2.1 we have $i(\gamma^2) + 2S_{P^2}^+(1) - \nu(\gamma^2) > n$ which contradicts to (3.11). So by Remark 2.1, we have

$$P^2 \approx I_2^{\diamond p_1} \diamond N_1(1, -1)^{\diamond p_2} \diamond R(\theta_1) \diamond R(\theta_2) \diamond \cdots \diamond R(\theta_{p_3}), \tag{3.14}$$

where $p_i \geq 0$ for $1 \leq i \leq 3$, $p_1 + p_2 + p_3 = n$ and $\theta_j \in (0, 2\pi)$ for $1 \leq j \leq p_3$.

Note that, since $\gamma^2(t) = \gamma(t - \tau)\gamma(\tau)$, we have

$$\gamma^2(2\tau) = \gamma(\tau)^2 = P^2. \tag{3.15}$$

By Definition 2.5 we have

$$\gamma^2(2\tau) = N\gamma(\tau)^{-1}N\gamma(\tau) = NP^{-1}NP. \tag{3.16}$$

So by (3.15) and (3.16) we have

$$P^2 = NP^{-1}NP. \tag{3.17}$$

By (3.17), Lemma 2.4, and $P \sim (-I_2) \diamond Q$ we have

$$\begin{aligned}
P^2 &= NP^{-1}NP \\
&\approx N((-I_2) \diamond Q)^{-1}N((-I_2) \diamond Q) \\
&= I_2 \diamond (N_{n-1}Q^{-1}N_{n-1}Q).
\end{aligned} \tag{3.18}$$

So by (3.14), we have

$$p_1 \geq 1. \tag{3.19}$$

Also by (3.18) and Lemma 2.5, we have

$$P^2 \approx I_2 \diamond (N_{n-1}Q'^{-1}N_{n-1}Q'), \quad \forall Q' \sim Q \text{ where } Q' \in \text{Sp}(2n-2). \tag{3.20}$$

By (3.14) it is easy to check that

$$\text{tr}(P^2) = 2n - 2p_3 + 2 \sum_{j=1}^{p_3} \cos \theta_j. \tag{3.21}$$

By (3.11), (3.14) and Lemma 2.1 we have

$$n = i(\gamma^2) + 2S_{P^2}^+(1) - \nu(\gamma^2) = i(\gamma^2) - p_2 \geq i(\gamma^2) - n + 1.$$

So

$$i(\gamma^2) \leq 2n - 1. \tag{3.22}$$

By (3.7) we have

$$i(\gamma^2) = n + i_{L_0}(\gamma) + i_{L_1}(\gamma). \tag{3.23}$$

Since $i_{L_0}(\gamma) \geq 0$ and $i_{L_1}(\gamma) \geq 0$, we have $n \leq i(\gamma^2) \leq 2n - 1$. So we can divide the index $i(\gamma^2)$ into the following three cases.

Case I. $i(\gamma^2) = n$.

In this case, by (3.7), $i_{L_0}(\gamma) \geq 0$, and $i_{L_1}(\gamma) \geq 0$, we have

$$i_{L_0}(\gamma) = 0 = i_{L_1}(\gamma). \tag{3.24}$$

So by (3.13) we have

$$\nu_{L_0}(\gamma) - \nu_{L_1}(\gamma) = n - 1. \tag{3.25}$$

Since $\nu_{L_1}(\gamma) \geq 1$ and $\nu_{L_0}(\gamma) \leq n$, we have

$$\nu_{L_0}(\gamma) = n, \quad \nu_{L_1}(\gamma) = 1. \tag{3.26}$$

By (3.7) we have

$$\nu(\gamma^2) = \nu(P^2) = n + 1. \quad (3.27)$$

By (3.12), (3.24) and (3.26) we have

$$S_{P^2}^+(1) = \frac{1-n}{2} + n = \frac{1+n}{2} = p + 1. \quad (3.28)$$

So by (3.14), (3.27), (3.28), and Lemma 2.1 we have

$$P^2 \approx I_2^{\diamond(p+1)} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_p), \quad (3.29)$$

where $\theta_j \in (0, 2\pi)$. By (3.5) and (3.26) we have $B = 0$. By (3.18), (3.3), and (3.4), we have

$$\begin{aligned} P^2 &= NP^{-1}NP \approx I_2 \diamond (N_{n-1}Q^{-1}N_{n-1}Q) \\ &= I_2 \diamond \begin{pmatrix} D^T & 0 \\ C^T & A^T \end{pmatrix} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \\ &= I_2 \diamond \begin{pmatrix} D^T A & 0 \\ 2C^T A & AD^T \end{pmatrix} \\ &= I_2 \diamond \begin{pmatrix} I_{2p} & 0 \\ 2A^T C & I_{2p} \end{pmatrix}. \end{aligned}$$

Hence $\sigma(P^2) = \{1\}$ which contradicts to (3.29) since $p \geq 1$.

Case II. $i(\gamma^2) = n + 2k$, where $1 \leq k \leq p$.

In this case by (3.7) we have

$$i_{L_0}(\gamma) + i_{L_1}(\gamma) = 2k.$$

Since $i_{L_0}(\gamma) \geq 0$ and $i_{L_1}(\gamma) \geq 0$ we can write $i_{L_0}(\gamma) = k + r$ and $i_{L_1}(\gamma) = k - r$ for some integer $-k \leq r \leq k$. Then by (3.13) we have

$$n - 1 \geq \nu_{L_0}(\gamma) - \nu_{L_1}(\gamma) = n - 2r - 1. \quad (3.30)$$

Thus $r \geq 0$ and $0 \leq r \leq k$.

By Theorem 2.1 and (i) of Lemma 2.5 we have

$$2r = i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2}M_\varepsilon(P) \leq n - \nu_{L_0}(P) \quad (3.31)$$

which yields that $\nu_{L_0}(\gamma) \leq n - 2r$. So by (3.30) and $\nu_{L_1}(\gamma) \geq 1$ we have

$$\nu_{L_0}(\gamma) = n - 2r, \quad \nu_{L_1}(\gamma) = 1. \quad (3.32)$$

Then by (3.12) we have

$$S_{P^2}^+(1) = (n - 2r) + \frac{1 - n}{2} - (k - r) = \frac{1 + n}{2} - k - r = p + 1 - k - r. \quad (3.33)$$

Then by (3.14) and $\nu(P^2) = n - 2r + 1$ and Lemma 2.1 we have

$$P^2 \approx I_2^{\diamond(p+1-k-r)} \diamond N_1(1, -1)^{\diamond 2k} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_q), \quad (3.34)$$

where $q = n - (p + 1 - k - r) - 2k = p + r - k \geq 0$. Then we have the following three subcases (i)-(iii).

(i) $q = 0$.

The only possibility is $k = p$ and $r = 0$, in this case $P^2 \approx I_2 \diamond N_1(1, -1)^{\diamond 2p}$ and $B = 0$. By direct computation we have

$$N_1(1, -1)^{\diamond 2p} \approx N_{2p}Q^{-1}N_{2p}Q = \begin{pmatrix} I_{n-1} & 0 \\ 2A^TC & I_{n-1} \end{pmatrix}. \quad (3.35)$$

Then by Lemma 2.3 we have

$$m^+(A^TC) = 2p.$$

By (ii) of Lemma 2.5 we have

$$\frac{1}{2}\text{sgn}M_\varepsilon(Q) \leq 2p - 2p = 0, \quad 0 < -\varepsilon \ll 1. \quad (3.36)$$

Thus by (3.36) and Theorem 2.1, for $0 < -\varepsilon \ll 1$ we have,

$$\begin{aligned} & (i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma)) \\ &= \frac{1}{2}\text{sgn}M_\varepsilon(P) \\ &= \frac{1}{2}\text{sgn}M_\varepsilon(I_2) + \frac{1}{2}M_\varepsilon(Q) \\ &= 0 + \frac{1}{2}M_\varepsilon(Q) \\ &\leq 0 \end{aligned}$$

which contradicts (3.13).

(ii) $q > 0$ and $r = 0$.

In this case $\nu_{L_0}(\gamma) = n$ and $\nu_{L_1}(\gamma) = 1$, also we have $B = 0$. By the equality of (3.35) we have

$$\text{tr}(P^2) = 2n$$

which contradicts to (3.21) with $p_3 = q > 0$.

(iii) $q > 0$ and $r > 0$.

In this case, by (3.33) we have $r < p$ (otherwise, then $p = r = k$. From (3.19) there holds $S_{P_2}^+(1) \geq 1$, so from (3.33) we have $1 \leq S_{P_2}^+(1) = 1 - p \leq 0$ a contradiction). Here it is easy to see $\text{rank} B = 2r$. Then there are two invertible $2p \times 2p$ matrices U and V with $\det U > 0$ and $\det V > 0$ such that

$$UBV = \begin{pmatrix} I_{2r} & 0 \\ 0 & 0 \end{pmatrix}.$$

So there holds

$$Q \sim \text{diag}(U, (U^T)^{-1}) Q \text{diag}((V^T)^{-1}, V) = \begin{pmatrix} A_1 & B_1 & I_{2r} & 0 \\ C_1 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & B_2 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix} := Q_1, \quad (3.37)$$

where for $j = 1, 2, 3$, A_j is a $2r \times 2r$ matrix, D_j is a $(2p - 2r) \times (2p - 2r)$ matrix for $j = 1, 2, 3$, B_j is a $2r \times (2p - 2r)$ matrix, and C_j is $(2p - 2r) \times 2r$ matrix. Since Q_1 is still a symplectic matrix, we have $Q_1^T J_{2p} Q_1 = J_{2p}$, then it is easy to check that

$$C_1 = 0, B_2 = 0. \quad (3.38)$$

So

$$Q_1 = \begin{pmatrix} A_1 & B_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix}. \quad (3.39)$$

So for the case (iii) of Case II, we have the following 3 subcases 1-3.

Subcase 1. $A_3 = 0$.

In this case since Q_1 is symplectic, by direct computation we have

$$N_{2p} Q_1^{-1} N_{2p} Q_1 = \begin{pmatrix} I_{2r} & * & * & * \\ * & I_{2p-2r} & * & * \\ * & * & I_{2r} & * \\ * & * & * & I_{2p-2r} \end{pmatrix}.$$

Hence we have

$$\text{tr}(N_{2p} Q_1^{-1} N_{2p} Q_1) = 4p.$$

Since $Q_1 \sim Q$, we have

$$P \sim (-I_2) \diamond Q_1. \quad (3.40)$$

Then by the proof of Lemma 2.4 we have

$$\begin{aligned} \text{tr} P^2 &= \text{tr}(NP^{-1}NP) \\ &= \text{tr} N((-I_2) \diamond Q_1)^{-1} N((-I_2) \diamond Q_1) \\ &= \text{tr} I_2 \diamond ((N_{2p} Q_1^{-1} N_{2p} Q_1)) \\ &= 4p + 2 = 2n. \end{aligned} \quad (3.41)$$

By (3.21) and $p_3 = q > 0$ we have

$$\text{tr}(P^2) < 2n. \quad (3.42)$$

(3.41) and (3.42) yield a contradiction.

Subcase 2. A_3 is invertible.

By Q_1 is symplectic we have

$$\begin{pmatrix} A_1^T & 0 \\ B_1^T & D_1^T \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ C_2 & D_2 \end{pmatrix} - \begin{pmatrix} A_3^T & C_3^T \\ B_3^T & D_3^T \end{pmatrix} \begin{pmatrix} I_{2r} & 0 \\ 0 & 0 \end{pmatrix} = I_{2p}. \quad (3.43)$$

Hence

$$D_1^T D_2 = I_{2p-2r}. \quad (3.44)$$

By direct computation we have

$$\begin{pmatrix} A_1 & B_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix} \begin{pmatrix} I_{2r} & -A_3^{-1}B_3 & 0 & 0 \\ 0 & I_{2p-2r} & 0 & 0 \\ 0 & 0 & I_{2r} & 0 \\ 0 & 0 & B_3^T(A_3^T)^{-1} & I_{2p-2r} \end{pmatrix} = \begin{pmatrix} A_1 & \tilde{B}_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & 0 & A_2 & 0 \\ C_3 & \tilde{D}_3 & \tilde{C}_2 & D_2 \end{pmatrix}.$$

So by (3.44) we have

$$\begin{aligned} & \begin{pmatrix} I_{2r} & -\tilde{B}_1 D_2^T & 0 & 0 \\ 0 & I_{2p-2r} & 0 & 0 \\ 0 & 0 & I_{2r} & 0 \\ 0 & 0 & D_2 \tilde{B}_1^T & I_{2p-2r} \end{pmatrix} \begin{pmatrix} A_1 & \tilde{B}_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & 0 & A_2 & 0 \\ C_3 & \tilde{D}_3 & \tilde{C}_2 & D_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 & 0 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & 0 & A_2 & 0 \\ \tilde{C}_3 & \tilde{D}_3 & \hat{C}_2 & D_2 \end{pmatrix} := Q_2. \end{aligned}$$

Then we have

$$Q_2 \sim Q_1 \sim Q. \quad (3.45)$$

Since Q_2 is a symplectic matrix, we have $Q_2^T J_{2p} Q_2 = J_{2p}$, then it is easy to check that

$$\tilde{C}_3 = 0, \hat{C}_2 = 0. \quad (3.46)$$

Hence we have

$$Q_2 = \begin{pmatrix} A_1 & I_{2r} \\ A_3 & A_2 \end{pmatrix} \diamond \begin{pmatrix} D_1 & 0 \\ \tilde{D}_3 & D_2 \end{pmatrix}. \quad (3.47)$$

Since

$$N_{2p-2r} \begin{pmatrix} D_1 & 0 \\ \tilde{D}_3 & D_2 \end{pmatrix}^{-1} N_{2p-2r} \begin{pmatrix} D_1 & 0 \\ \tilde{D}_3 & D_2 \end{pmatrix} = \begin{pmatrix} I_{2p-2r} & 0 \\ 2D_1^T \tilde{D}_3 & I_{2p-2r} \end{pmatrix}, \quad (3.48)$$

by (3.45), (3.20), and Lemma 2.4, there is a symplectic matrix W such that

$$P^2 \approx I_2 \diamond W \diamond \begin{pmatrix} I_{2p-2r} & 0 \\ 2D_1^T \tilde{D}_3 & I_{2p-2r} \end{pmatrix}. \quad (3.49)$$

Then by (3.14) and Lemma 2.3, $D_1^T \tilde{D}_3$ is semipositive and

$$1 + m^0(D_1^T \tilde{D}_3) \leq S_{p^2}^+(1).$$

So by (3.33) we have

$$m^0(D_1^T \tilde{D}_3) \leq p + 1 - k - r - 1 = p - k - r = (2p - 2r) - (p + k - r) \leq 2p - 2r - 1. \quad (3.50)$$

Since $D_1^T \tilde{D}_3$ is a semipositive $(2p - 2r) \times (2p - 2r)$ matrix, by (3.50) we have $m^+(D_1^T \tilde{D}_3) > 0$. Then by Theorem 2.1, (ii) of Lemma 2.5 and Lemma 2.6, for $0 < -\varepsilon \ll 1$ we have

$$\begin{aligned} & (i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma)) \\ &= \frac{1}{2} \left(M_\varepsilon(-I_2) + M_\varepsilon \left(\begin{pmatrix} A_1 & I_{2r} \\ A_3 & A_2 \end{pmatrix} \right) + M_\varepsilon \left(\begin{pmatrix} D_1 & 0 \\ \tilde{D}_3 & D_2 \end{pmatrix} \right) \right) \\ &\leq \frac{1}{2} (0 + 4r + 2(2p - 2r - 1)) \\ &= 2p - 1 \\ &= n - 2 \end{aligned} \quad (3.51)$$

which contradicts to (3.13).

Subcase 3. $A_3 \neq 0$ and A_3 is not invertible.

In this case, suppose $\text{rank} A_3 = \lambda$, then $0 < \lambda < 2r$. There is a invertible $2r \times 2r$ matrix G with $\det G > 0$ such that

$$GA_3G^{-1} = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.52)$$

where Λ is a $\lambda \times \lambda$ invertible matrix. Then we have

$$\begin{aligned} & \begin{pmatrix} (G^T)^{-1} & 0 & 0 & 0 \\ 0 & I_{2p-2r} & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & I_{2p-2r} \end{pmatrix} \begin{pmatrix} A_1 & B_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix} \begin{pmatrix} (G)^{-1} & 0 & 0 & 0 \\ 0 & I_{2p-2r} & 0 & 0 \\ 0 & 0 & G^T & 0 \\ 0 & 0 & 0 & I_{2p-2r} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{A}_1 & \tilde{B}_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ GA_3G^{-1} & \tilde{B}_3 & \tilde{A}_2 & 0 \\ \tilde{C}_3 & D_3 & \tilde{C}_2 & D_2 \end{pmatrix} := Q_3. \end{aligned} \quad (3.53)$$

By (3.52) we can write Q_3 as the following block form

$$Q_3 = \begin{pmatrix} U_1 & U_2 & F_1 & I_\lambda & 0 & 0 \\ U_3 & U_4 & F_2 & 0 & I_{2r-\lambda} & 0 \\ 0 & 0 & D_1 & 0 & 0 & 0 \\ \Lambda & 0 & E_1 & W_1 & W_2 & 0 \\ 0 & 0 & E_2 & W_3 & W_4 & 0 \\ G_1 & G_2 & D_3 & K_1 & K_2 & D_2 \end{pmatrix}. \quad (3.54)$$

$$\text{Let } R_1 = \begin{pmatrix} I_\lambda & 0 & 0 \\ 0 & I_{2r-\lambda} & 0 \\ -G_1\Lambda^{-1} & 0 & I_{2p-2r} \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} I_\lambda & 0 & -\Lambda^{-1}E_1 \\ 0 & I_{2r-\lambda} & 0 \\ 0 & 0 & I_{2p-2r} \end{pmatrix}. \text{ By (3.54) we have}$$

$$\text{diag}((R_1^T)^{-1}, R_1)Q_3\text{diag}(R_2, (R_2^T)^{-1}) = \begin{pmatrix} U_1 & U_2 & \tilde{F}_1 & I_\lambda & 0 & 0 \\ U_3 & U_4 & \tilde{F}_2 & 0 & I_{2r-\lambda} & 0 \\ 0 & 0 & D_1 & 0 & 0 & 0 \\ \Lambda & 0 & 0 & W_1 & W_2 & 0 \\ 0 & 0 & E_2 & W_3 & W_4 & 0 \\ 0 & G_2 & \tilde{D}_3 & \tilde{K}_1 & \tilde{K}_2 & D_2 \end{pmatrix} := Q_4.$$

Since Q_4 is a symplectic matrix we have

$$Q_4^T J Q_4 = J.$$

Then by (3.55) and direct computation we have $U_2 = 0$, $U_3 = 0$, $W_2 = 0$, $W_3 = 0$, $\tilde{F}_1 = 0$, $\tilde{K}_1 = 0$, and U_1 , U_4 , W_1 , W_4 are all symmetric matrices, and

$$U_4 W_4 = I_{2r-\lambda}, \quad (3.55)$$

$$D_1 D_2^T = I_{2p-2r}, \quad (3.56)$$

$$U_4 \tilde{E}_2 = G_2^T D_1, \quad (3.57)$$

So

$$Q_4 = \begin{pmatrix} U_1 & 0 & 0 & I_\lambda & 0 & 0 \\ 0 & U_4 & \tilde{F}_2 & 0 & I_{2r-\lambda} & 0 \\ 0 & 0 & D_1 & 0 & 0 & 0 \\ \Lambda & 0 & 0 & W_1 & 0 & 0 \\ 0 & 0 & \tilde{E}_2 & 0 & W_4 & 0 \\ 0 & G_2 & \tilde{D}_3 & 0 & K_2 & D_2 \end{pmatrix}. \quad (3.58)$$

By (3.55)-(3.57), we have both \tilde{E}_2 and G_2 are zero or nonzero. By definition 2.3 we have $Q_4 \sim Q_3 \sim Q$. Then by (3.32), $\begin{pmatrix} \Lambda & 0 & 0 \\ 0 & 0 & \tilde{E}_2 \\ 0 & G_2 & \tilde{D}_3 \end{pmatrix}$ is invertible. So both \tilde{E}_2 and G_2 are nonzero.

Since Q_4 is symplectic, by (3.57) we have

$$\begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_4 & \tilde{F}_2 \\ 0 & 0 & D_1 \end{pmatrix}^T \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & 0 & \tilde{E}_2 \\ 0 & G_2 & \tilde{D}_3 \end{pmatrix} = \begin{pmatrix} U_1 \Lambda & 0 & 0 \\ 0 & 0 & U_4 \tilde{E}_2 \\ 0 & (U_4 \tilde{E}_2)^T & D_1^T \tilde{D}_3 + \tilde{B}_2^T \tilde{E}_2 \end{pmatrix} \quad (3.59)$$

which is a symmetric matrix.

Denote by $F = \begin{pmatrix} 0 & U_4 \tilde{E}_2 \\ (U_4 \tilde{E}_2)^T & D_1^T \tilde{D}_3 + \tilde{B}_2^T \tilde{E}_2 \end{pmatrix}$. Since $U_4 \tilde{E}_2$ is nonzero, in the following we prove that $m^+(F) \geq 1$.

Note that here $U_4 \tilde{E}_2$ is a $(2r-\lambda) \times (2p-2r)$ matrix and $D_1^T \tilde{D}_3 + \tilde{B}_2^T \tilde{E}_2$ is a $(2p-2r) \times (2p-2r)$ matrix. Denote by $U_4 \tilde{E}_2 = (e_{ij})$ and $D_1^T \tilde{D}_3 + \tilde{B}_2^T \tilde{E}_2 = (d_{ij})$, where e_{ij} and d_{ij} are elements on the i -th row and j -th column of the corresponding matrix. Since $U_4 \tilde{E}_2$ is nonzero, there exist an $e_{ij} \neq 0$ for some $1 \leq i \leq 2r-\lambda$ and $1 \leq j \leq 2p-2r$. Let $x = (0, \dots, 0, e_{ij}, 0, \dots, 0)^T \in \mathbf{R}^{2r-\lambda}$ whose i -th row is e_{ij} and other rows are all zero, and $y = (0, \dots, 0, \rho, 0, \dots, 0)^T \in \mathbf{R}^{2p-2r}$ whose j -th row is ρ and other rows are all zero. Then we have

$$F \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2\rho e_{ij}^2 - \rho^2 d_{jj} > 0$$

for $\rho > 0$ is small enough. Hence the dimension of positive definite space of F is at least 1, thus $m^+(F) \geq 1$. Then

$$m^+ \left(\begin{pmatrix} U_1 \Lambda & 0 & 0 \\ 0 & 0 & U_4 \tilde{E}_2 \\ 0 & (U_4 \tilde{E}_2)^T & D_1^T \tilde{D}_3 + \tilde{B}_2^T \tilde{E}_2 \end{pmatrix} \right) = m^+(\Lambda) + m^+(F) \geq 1. \quad (3.60)$$

Then by (3.59), (3.60) and (ii) of Lemma 2.5, we have

$$\frac{1}{2} \text{sgn} M_\varepsilon(Q_4) \leq 2p - 1 = n - 2, \quad 0 < -\varepsilon \ll 1. \quad (3.61)$$

Since $Q \sim Q_4$, by (3.61) and Lemma 2.4 we have

$$\frac{1}{2} \text{sgn} M_\varepsilon(Q) \leq 2p - 1, \quad 0 < -\varepsilon \ll 1. \quad (3.62)$$

Then since $P \sim (-I_2) \diamond Q$, by Theorem 2.1, Remark 2.2 and Lemma 2.4 we have

$$\begin{aligned} & (i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma)) \\ &= \frac{1}{2} M_\varepsilon(P) \\ &= \frac{1}{2} \text{sgn} M_\varepsilon((-I_2) \diamond Q) \\ &= \frac{1}{2} \text{sgn} M_\varepsilon(-I_2) + \frac{1}{2} \text{sgn} M_\varepsilon(Q) \\ &= 0 + \frac{1}{2} \text{sgn} M_\varepsilon(Q) \\ &\leq n - 2. \end{aligned} \quad (3.63)$$

Thus (3.13) and (3.63) yields a contradiction. And in Case II we can always obtain a contradiction.

Case III. $i(\gamma^2) = n + 2k + 1$, where $0 \leq k \leq p - 1$.

In this case by (3.7) we have

$$i_{L_0}(\gamma) + i_{L_1}(\gamma) = 2k + 1. \quad (3.64)$$

Since $i_{L_0}(\gamma) \geq 0$ and $i_{L_1}(\gamma) \geq 0$ we can write $i_{L_0}(\gamma) = k + 1 + r$ and $i_{L_1}(\gamma) = k - r$ for some integer $-k \leq r \leq k$. Then by (3.13) we have

$$n - 1 \geq \nu_{L_0}(\gamma) - \nu_{L_1}(\gamma) = n - 2r - 2. \quad (3.65)$$

Thus $r \geq 0$ and $0 \leq r \leq k$.

By Theorem 2.1 and (i) of Lemma 2.5 we have

$$2r + 1 = i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2} M_\varepsilon(P) \leq n - \nu_{L_0}(\gamma) \quad (3.66)$$

which yields $\nu_{L_0}(\gamma) \leq n - 2r - 1$. Then by (3.65) and $\nu_{L_1}(\gamma) \geq 1$ we have

$$\nu_{L_0}(\gamma) = n - 2r - 1, \quad \nu_{L_1}(\gamma) = 1. \quad (3.67)$$

Then by (3.12) we have

$$S_{P^2}^+(1) = (n - 2r - 1) + \frac{1 - n}{2} - (k - r) = \frac{1 + n}{2} - k - r - 1 = p - k - r \geq 1. \quad (3.68)$$

Then by (3.14) and $\nu(P^2) = \nu_{L_0}(\gamma) + \nu_{L_1}(\gamma) = n - 2r$ and Lemma 2.1 we have

$$P^2 \approx I_2^{\diamond(p-k-r)} \diamond N_1(1, -1)^{\diamond(2k+1)} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_q),$$

where $q = n - (p - k - r) - (2k + 1) = p + r - k \geq p - k \geq 1$.

Since in this case $\text{rank} B = 2r + 1 \leq n - 2$, by the same argument of (iii) in Case II, we have

$$Q \sim Q_1 = \begin{pmatrix} A_1 & B_1 & I_{2r+1} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix}.$$

Then by the same argument of Subcases 1, 2, 3 of Case II, we can always obtain a contradiction in Case III. The proof of Theorem 3.1 is complete. \blacksquare

Now we are ready to give a proof of Theorem 1.1. For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, let $j_\Sigma : \Sigma \rightarrow [0, +\infty)$ be the gauge function of Σ defined by

$$j_\Sigma(0) = 0, \quad \text{and} \quad j_\Sigma(x) = \inf\{\lambda > 0 \mid \frac{x}{\lambda} \in C\}, \quad \forall x \in \mathbf{R}^{2n} \setminus \{0\},$$

where C is the domain enclosed by Σ .

Define

$$H_\alpha(x) = (j_\Sigma(x))^\alpha, \quad \alpha > 1, \quad H_\Sigma(x) = H_2(x), \quad \forall x \in \mathbf{R}^{2n}. \quad (3.69)$$

Then $H_\Sigma \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^{1,1}(\mathbf{R}^{2n}, \mathbf{R})$.

We consider the following fixed energy problem

$$\dot{x}(t) = JH'_\Sigma(x(t)), \quad (3.70)$$

$$H_\Sigma(x(t)) = 1, \quad (3.71)$$

$$x(-t) = Nx(t), \quad (3.72)$$

$$x(\tau + t) = x(t), \quad \forall t \in \mathbf{R}. \quad (3.73)$$

Denote by $\mathcal{J}_b(\Sigma, 2)$ ($\mathcal{J}_b(\Sigma, \alpha)$ for $\alpha = 2$ in (3.69)) the set of all solutions (τ, x) of problem (3.70)-(3.73) and by $\tilde{\mathcal{J}}_b(\Sigma, 2)$ the set of all geometrically distinct solutions of (3.70)-(3.73). By Remark 1.2 of [14] or discussion in [17], elements in $\mathcal{J}_b(\Sigma)$ and $\mathcal{J}_b(\Sigma, 2)$ are one to one correspondent. So we have $\# \tilde{\mathcal{J}}_b(\Sigma) = \# \tilde{\mathcal{J}}_b(\Sigma, 2)$.

For readers' convenience in the following we list some known results which will be used in the proof of Theorem 1.1. In the following of this paper, we write $(i_{L_0}(\gamma, k), \nu_{L_0}(\gamma, k)) = (i_{L_0}(\gamma^k), \nu_{L_0}(\gamma^k))$ for any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $k \in \mathbf{N}$, where γ^k is defined by Definition 2.5. We have

Lemma 3.1. (Theorem 1.5 and of [14] and Theorem 4.3 of [18]) *Let $\gamma_j \in \mathcal{P}_{\tau_j}(2n)$ for $j = 1, \dots, q$. Let $M_j = \gamma_j^2(2\tau_j) = N\gamma_j(\tau_j)^{-1}N\gamma_j(\tau_j)$, for $j = 1, \dots, q$. Suppose*

$$\hat{i}_{L_0}(\gamma_j) > 0, \quad j = 1, \dots, q.$$

Then there exist infinitely many $(R, m_1, m_2, \dots, m_q) \in \mathbf{N}^{q+1}$ such that

- (i) $\nu_{L_0}(\gamma_j, 2m_j \pm 1) = \nu_{L_0}(\gamma_j)$,
- (ii) $i_{L_0}(\gamma_j, 2m_j - 1) + \nu_{L_0}(\gamma_j, 2m_j - 1) = R - (i_{L_1}(\gamma_j) + n + S_{M_j}^+(1) - \nu_{L_0}(\gamma_j))$,
- (iii) $i_{L_0}(\gamma_j, 2m_j + 1) = R + i_{L_0}(\gamma_j)$.

and (iv) $\nu(\gamma_j^2, 2m_j \pm 1) = \nu(\gamma_j^2)$,

- (v) $i(\gamma_j^2, 2m_j - 1) + \nu(\gamma_j^2, 2m_j - 1) = 2R - (i(\gamma_j^2) + 2S_{M_j}^+(1) - \nu(\gamma_j^2))$,
- (vi) $i(\gamma_j^2, 2m_j + 1) = 2R + i(\gamma_j^2)$,

where we have set $i(\gamma_j^2, n_j) = i(\gamma_j^{2n_j}, [0, 2n_j\tau_j])$, $\nu(\gamma_j^2, n_j) = \nu(\gamma_j^{2n_j}, [0, 2n_j\tau_j])$ for $n_j \in \mathbf{N}$.

Lemma 3.2 (Lemma 1.1 of [14]) *Let $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ be symmetric in the sense that $x(t + \frac{\tau}{2}) = -x(t)$ for all $t \in \mathbf{R}$ and γ be the associated symplectic path of (τ, x) . Set $M = \gamma(\frac{\tau}{2})$. Then there is a continuous symplectic path*

$$\Psi(s) = P(s)MP(s)^{-1}, \quad s \in [0, 1]$$

such that

$$\Psi(0) = M, \quad \Psi(1) = (-I_2) \diamond \tilde{M}, \quad \tilde{M} \in \text{Sp}(2n - 2),$$

$$\nu_1(\Psi(s)) = \nu_1(M), \quad \nu_2(\Psi(s)) = \nu_2(M), \quad \forall s \in [0, 1],$$

where $P(s) = \begin{pmatrix} \psi(s)^{-1} & 0 \\ 0 & \psi(s)^T \end{pmatrix}$ and ψ is a continuous $n \times n$ matrix path with $\det \psi(s) > 0$ for all $s \in [0, 1]$.

For any $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ and $m \in \mathbf{N}$, as in [14] we denote by $i_{L_j}(x, m) = i_{L_j}(\gamma_x^m, [0, \frac{m\tau}{2}])$ and $\nu_{L_j}(x, m) = \nu_{L_j}(\gamma_x^m, [0, \frac{m\tau}{2}])$ for $j = 0, 1$ respectively. Also we denote by $i(x, m) = i(\gamma_x^{2m}, [0, m\tau])$ and $\nu(x, m) = \nu(\gamma_x^{2m}, [0, m\tau])$. If $m = 1$, we denote by $i(x) = i(x, 1)$ and $\nu(x) = \nu(x, 1)$. By Lemma 6.3 of [14] we have

Lemma 3.3. *Suppose $\#\tilde{\mathcal{J}}_b(\Sigma) < +\infty$. Then there exist an integer $K \geq 0$ and an injection map $\phi : \mathbf{N} + K \mapsto \mathcal{J}_b(\Sigma, 2) \times \mathbf{N}$ such that*

(i) For any $k \in \mathbf{N} + K$, $[(\tau, x)] \in \mathcal{J}_b(\Sigma, 2)$ and $m \in \mathbf{N}$ satisfying $\phi(k) = ([(\tau, x)], m)$, there holds

$$i_{L_0}(x, m) \leq k - 1 \leq i_{L_0}(x, m) + \nu_{L_0}(x, m) - 1,$$

where x has minimal period τ .

(ii) For any $k_j \in \mathbf{N} + K$, $k_1 < k_2$, $(\tau_j, x_j) \in \mathcal{J}_b(\Sigma, 2)$ satisfying $\phi(k_j) = ([(\tau_j, x_j)], m_j)$ with $j = 1, 2$ and $[(\tau_1, x_1)] = [(\tau_2, x_2)]$, there holds

$$m_1 < m_2.$$

Lemma 3.4. (Lemma 7.2 of [14]) *Let $\gamma \in \mathcal{P}_\tau(2n)$ be extended to $[0, +\infty)$ by $\gamma(\tau + t) = \gamma(t)\gamma(\tau)$ for all $t > 0$. Suppose $\gamma(\tau) = M = P^{-1}(I_2 \diamond \tilde{M})P$ with $\tilde{M} \in \text{Sp}(2n - 2)$ and $i(\gamma) \geq n$. Then we have*

$$i(\gamma, 2) + 2S_{M^2}^+(1) - \nu(\gamma, 2) \geq n + 2.$$

Lemma 3.5 (Lemma 7.3 of [14]) *For any $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ and $m \in \mathbf{N}$, we have*

$$\begin{aligned} i_{L_0}(x, m+1) - i_{L_0}(x, m) &\geq 1, \\ i_{L_0}(x, m+1) + \nu_{L_0}(x, m+1) - 1 &\geq i_{L_0}(x, m+1) > i_{L_0}(x, m) + \nu_{L_0}(x, m) - 1. \end{aligned}$$

Proof of Theorem 1.1. By Theorem 1.1 of [14] we have $\#\tilde{\mathcal{J}}_b(\Sigma) \geq [\frac{n}{2}] + 1$ for $n \in \mathbf{N}$. So we only need to prove Theorem q.q for the case $n \geq 3$ and n is odd. The method of the proof is similar as that of [14].

It suffices to consider the case $\#\tilde{\mathcal{J}}_b(\Sigma) < +\infty$. Since $-\Sigma = \Sigma$, for $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ we have

$$\begin{aligned} H_\Sigma(x) &= H_\Sigma(-x), \\ H'_\Sigma(x) &= -H'_\Sigma(-x), \\ H''_\Sigma(x) &= H''_\Sigma(-x). \end{aligned} \tag{3.74}$$

So $(\tau, -x) \in \mathcal{J}_b(\Sigma, 2)$. By (3.74) and the definition of γ_x we have that

$$\gamma_x = \gamma_{-x}.$$

So we have

$$\begin{aligned} (i_{L_0}(x, m), \nu_{L_0}(x, m)) &= (i_{L_0}(-x, m), \nu_{L_0}(-x, m)), \\ (i_{L_1}(x, m), \nu_{L_1}(x, m)) &= (i_{L_1}(-x, m), \nu_{L_1}(-x, m)), \quad \forall m \in \mathbf{N}. \end{aligned} \quad (3.75)$$

So we can write

$$\tilde{\mathcal{J}}_b(\Sigma, 2) = \{[(\tau_j, x_j)] | j = 1, \dots, p\} \cup \{[(\tau_k, x_k)], [(\tau_k, -x_k)] | k = p+1, \dots, p+q\}. \quad (3.76)$$

with $x_j(\mathbf{R}) = -x_j(\mathbf{R})$ for $j = 1, \dots, p$ and $x_k(\mathbf{R}) \neq -x_k(\mathbf{R})$ for $k = p+1, \dots, p+q$. Here we remind that (τ_j, x_j) has minimal period τ_j for $j = 1, \dots, p+q$ and $x_j(\frac{\tau_j}{2} + t) = -x_j(t)$, $t \in \mathbf{R}$ for $j = 1, \dots, p$.

By Lemma 3.3 we have an integer $K \geq 0$ and an injection map $\phi : \mathbf{N} + K \rightarrow \mathcal{J}_b(\Sigma, 2) \times \mathbf{N}$. By (3.75), (τ_k, x_k) and $(\tau_k, -x_k)$ have the same (i_{L_0}, ν_{L_0}) -indices. So by Lemma 3.3, without loss of generality, we can further require that

$$\text{Im}(\phi) \subseteq \{[(\tau_k, x_k)] | k = 1, 2, \dots, p+q\} \times \mathbf{N}. \quad (3.77)$$

By the strict convexity of H_Σ and (6.19) of [14]), we have

$$\hat{i}_{L_0}(x_k) > 0, \quad k = 1, 2, \dots, p+q.$$

Applying Lemma 3.1 to the following associated symplectic paths

$$\gamma_1, \dots, \gamma_{p+q}, \gamma_{p+q+1}, \dots, \gamma_{p+2q}$$

of $(\tau_1, x_1), \dots, (\tau_{p+q}, x_{p+q}), (2\tau_{p+1}, x_{p+1}^2), \dots, (2\tau_{p+q}, x_{p+q}^2)$ respectively, there exists a vector $(R, m_1, \dots, m_{p+2q}) \in \mathbf{N}^{p+2q+1}$ such that $R > K + n$ and

$$i_{L_0}(x_k, 2m_k + 1) = R + i_{L_0}(x_k), \quad (3.78)$$

$$\begin{aligned} & i_{L_0}(x_k, 2m_k - 1) + \nu_{L_0}(x_k, 2m_k - 1) \\ &= R - (i_{L_1}(x_k) + n + S_{M_k}^+(1) - \nu_{L_0}(x_k)), \end{aligned} \quad (3.79)$$

for $k = 1, \dots, p+q$, $M_k = \gamma_k^2(\tau_k)$, and

$$i_{L_0}(x_k, 4m_k + 2) = R + i_{L_0}(x_k, 2), \quad (3.80)$$

$$\begin{aligned} & i_{L_0}(x_k, 4m_k - 2) + \nu_{L_0}(x_k, 4m_k - 2) \\ &= R - (i_{L_1}(x_k, 2) + n + S_{M_k}^+(1) - \nu_{L_0}(x_k, 2)), \end{aligned} \quad (3.81)$$

for $k = p + q + 1, \dots, p + 2q$ and $M_k = \gamma_k^4(2\tau_k) = \gamma_k^2(\tau_k)^2$.

By Lemma 3.1, we also have

$$i(x_k, 2m_k + 1) = 2R + i(x_k), \quad (3.82)$$

$$i(x_k, 2m_k - 1) + \nu(x_k, 2m_k - 1) = 2R - (i(x_k) + 2S_{M_k}^+(1) - \nu(x_k)), \quad (3.83)$$

for $k = 1, \dots, p + q$, $M_k = \gamma_k^2(\tau_k)$, and

$$i(x_k, 4m_k + 2) = 2R + i(x_k, 2), \quad (3.84)$$

$$i(x_k, 4m_k - 2) + \nu(x_k, 4m_k - 2) = 2R - (i(x_k, 2) + 2S_{M_k}^+(1) - \nu(x_k, 2)), \quad (3.85)$$

for $k = p + q + 1, \dots, p + 2q$ and $M_k = \gamma_k^4(2\tau_k) = \gamma_k^2(\tau_k)^2$.

From (3.77), we can set

$$\phi(R - (s - 1)) = ([(\tau_{k(s)}, x_{k(s)}], m(s)), \quad \forall s \in S := \left\{1, 2, \dots, \left\lceil \frac{n+1}{2} \right\rceil + 1\right\},$$

where $k(s) \in \{1, 2, \dots, p + q\}$ and $m(s) \in \mathbf{N}$.

We continue our proof to study the symmetric and asymmetric orbits separately. Let

$$S_1 = \{s \in S \mid k(s) \leq p\}, \quad S_2 = S \setminus S_1.$$

We shall prove that $\#S_1 \leq p$ and $\#S_2 \leq 2q$, together with the definitions of S_1 and S_2 , these yield Theorem 1.1.

Claim 1. $\#S_1 \leq p$.

Proof of Claim 1. By the definition of S_1 , $([(\tau_{k(s)}, x_{k(s)}], m(s))$ is symmetric when $k(s) \leq p$. We further prove that $m(s) = 2m_{k(s)}$ for $s \in S_1$.

In fact, by the definition of ϕ and Lemma 3.3, for all $s = 1, 2, \dots, \left\lceil \frac{n+1}{2} \right\rceil + 1$ we have

$$\begin{aligned} i_{L_0}(x_{k(s)}, m(s)) &\leq (R - (s - 1)) - 1 = R - s \\ &\leq i_{L_0}(x_{k(s)}, m(s)) + \nu_{L_0}(x_{k(s)}, m(s)) - 1. \end{aligned} \quad (3.86)$$

By the strict convexity of H_Σ and Lemma 2.2, we have $i_{L_0}(x_{k(s)}) \geq 0$, so there holds

$$i_{L_0}(x_{k(s)}, m(s)) \leq R - s < R \leq R + i_{L_0}(x_{k(s)}) = i_{L_0}(x_{k(s)}, 2m_{k(s)} + 1), \quad (3.87)$$

for every $s = 1, 2, \dots, \left\lceil \frac{n+1}{2} \right\rceil + 1$, where we have used (3.78) in the last equality. Note that the proofs of (3.86) and (3.87) do not depend on the condition $s \in S_1$.

By Lemma 3.2, γ_{x_k} satisfies conditions of Theorem 3.1 with $\tau = \frac{\tau_k}{2}$. Note that by definition $i_{L_1}(x_k) = i_{L_1}(\gamma_{x_k})$ and $\nu_{L_0}(x_k) = \nu_{L_0}(\gamma_{x_k})$. So by Theorem 3.1 we have

$$i_{L_1}(x_k) + S_{M_k}^+(1) - \nu_{L_0}(x_k) > \frac{1-n}{2}, \quad \forall k = 1, \dots, p. \quad (3.88)$$

Also for $1 \leq s \leq \left\lceil \frac{n+1}{2} \right\rceil + 1$, we have

$$-\frac{n+3}{2} = -\left(\left\lceil \frac{n+1}{2} \right\rceil + 1\right) \leq -s. \quad (3.89)$$

Hence by (3.86), (3.88) and (3.89), if $k(s) \leq p$ we have

$$\begin{aligned} & i_{L_0}(x_{k(s)}, 2m_{k(s)} - 1) + \nu_{L_0}(x_{k(s)}, 2m_{k(s)} - 1) - 1 \\ &= R - (i_{L_1}(x_{k(s)}) + n + S_{M_{k(s)}}^+(1) - \nu_{L_0}(x_{k(s)})) - 1 \\ &< R - \frac{1-n}{2} - 1 - n = R - \frac{n+3}{2} \leq R - s \\ &\leq i_{L_0}(x_{k(s)}, m(s)) + \nu_{L_0}(x_{k(s)}, m(s)) - 1. \end{aligned} \quad (3.90)$$

Thus by (3.87) and (3.90) and Lemma 3.5 of [14] we have

$$2m_{k(s)} - 1 < m(s) < 2m_{k(s)} + 1. \quad (3.91)$$

Hence

$$m(s) = 2m_{k(s)}. \quad (3.92)$$

So we have

$$\phi(R - s + 1) = ([(\tau_{k(s)}, x_{k(s)})], 2m_{k(s)}), \quad \forall s \in S_1. \quad (3.93)$$

Then by the injectivity of ϕ , it induces another injection map

$$\phi_1 : S_1 \rightarrow \{1, \dots, p\}, \quad s \mapsto k(s). \quad (3.94)$$

There for $\#S_1 \leq p$. Claim 1 is proved.

Claim 2. $\#S_2 \leq 2q$.

Proof of Claim 2. By the formulas (3.82)-(3.85), and (59) of [13] (also Claim 4 on p. 352 of [16]), we have

$$m_k = 2m_{k+q} \quad \text{for } k = p+1, p+2, \dots, p+q. \quad (3.95)$$

We set $\mathcal{A}_k = i_{L_1}(x_k, 2) + S_{M_k}^+(1) - \nu_{L_0}(x_k, 2)$ and $\mathcal{B}_k = i_{L_0}(x_k, 2) + S_{M_k}^+(1) - \nu_{L_1}(x_k, 2)$, $p+1 \leq k \leq p+q$, where $M_k = \gamma_k(2\tau_k) = \gamma(\tau_k)^2$. By (3.7), we have

$$\mathcal{A}_k + \mathcal{B}_k = i(x_k, 2) + 2S_{M_k}^+(1) - \nu(x_k, 2) - n, \quad p+1 \leq k \leq p+q. \quad (3.96)$$

By similar discussion of the proof of Lemma 3.2, for any $p+1 \leq k \leq p+q$ there exist $P_k \in \text{Sp}(2n)$ and $\tilde{M}_k \in \text{Sp}(2n-2)$ such that

$$\gamma(\tau_k) = P_k^{-1}(I_2 \diamond \tilde{M}_k)P_k.$$

Hence by Lemma 3.4 and (3.96), we have

$$\mathcal{A}_k + \mathcal{B}_k \geq n + 2 - n = 2. \quad (3.97)$$

By Theorem 2.1, there holds

$$|\mathcal{A}_k - \mathcal{B}_k| = |(i_{L_0}(x_k, 2) + \nu_{L_0}(x_k, 2)) - (i_{L_1}(x_k, 2) + \nu_{L_1}(x_k, 2))| \leq n. \quad (3.98)$$

So by (3.97) and (3.98) we have

$$\mathcal{A}_k \geq \frac{1}{2}((\mathcal{A}_k + \mathcal{B}_k) - |\mathcal{A}_k - \mathcal{B}_k|) \geq \frac{2-n}{2}, \quad p+1 \leq k \leq p+q. \quad (3.99)$$

By (3.81), (3.86), (3.89), (3.95) and (3.99), for $p+1 \leq k(s) \leq p+q$ we have

$$\begin{aligned} & i_{L_0}(x_{k(s)}, 2m_{k(s)} - 2) + \nu_{L_0}(x_{k(s)}, 2m_{k(s)} - 2) - 1 \\ &= i_{L_0}(x_{k(s)}, 4m_{k(s)+q} - 2) + \nu_{L_0}(x_{k(s)}, 4m_{k(s)+q} - 2) - 1 \\ &= R - (i_{L_1}(x_{k(s)}, 2) + n + S_{M_{k(s)}}^+(1) - \nu_{L_0}(x_{k(s)}, 2)) - 1 \\ &= R - \mathcal{A}_{k(s)} - 1 - n \\ &\leq R - \frac{2-n}{2} - 1 - n \\ &= R - (2 + \frac{n}{2}) \\ &< R - \frac{n+3}{2} \\ &\leq R - s \\ &\leq i_{L_0}(x_{k(s)}, m(s)) + \nu_{L_0}(x_{k(s)}, m(s)) - 1. \end{aligned} \quad (3.100)$$

Thus by (3.87), (3.100) and Lemma 3.5, we have

$$2m_{k(s)} - 2 < m(s) < 2m_{k(s)} + 1, \quad p < k(s) \leq p+q.$$

So

$$m(s) \in \{2m_{k(s)} - 1, 2m_{k(s)}\}, \quad \text{for } p < k(s) \leq p+q.$$

Especially this yields that for any s_0 and $s \in S_2$, if $k(s) = k(s_0)$, then

$$m(s) \in \{2m_{k(s)} - 1, 2m_{k(s)}\} = \{2m_{k(s_0)} - 1, 2m_{k(s_0)}\}.$$

Thus by the injectivity of the map ϕ from Lemma 3.3, we have

$$\#\{s \in S_2 | k(s) = k(s_0)\} \leq 2$$

which yields Claim 2.

By Claim 1 and Claim 2, we have

$$\#\tilde{\mathcal{J}}_b(\Sigma) = \#\tilde{\mathcal{J}}_b(\Sigma, 2) = p + 2q \geq \#S_1 + \#S_2 = \left\lfloor \frac{n+1}{2} \right\rfloor + 1.$$

The proof of Theorem 1.1 is complete. ■

Proof of Theorem 1.2. By [13], there are at least n closed characteristics on every C^2 compact convex central symmetric hypersurface Σ of \mathbf{R}^{2n} . Hence by Example 1.1 the assumption of Theorem 1.2 is reasonable. Here we prove the case $n = 5$, the proof of the case $n = 4$ is the same.

We call a closed characteristic x on Σ a *dual brake orbit* on Σ if $x(-t) = -Nx(t)$. Then by the similar proof of Lemma 3.1 of [22], a closed characteristic x on Σ can become a dual brake orbit after suitable time translation if and only if $x(\mathbf{R}) = -Nx(\mathbf{R})$. So by Lemma 3.1 of [22] again, if a closed characteristic x on Σ can both become brake orbits and dual brake orbits after suitable translation, then $x(\mathbf{R}) = Nx(\mathbf{R}) = -Nx(\mathbf{R})$, Thus $x(\mathbf{R}) = -x(\mathbf{R})$.

Since we also have $-N\Sigma = \Sigma$, $(-N)^2 = I_{2n}$ and $(-N)J = -J(-N)$, dually by the same proof of Theorem 1.1, there are at least $\lfloor (n+1)/2 \rfloor + 1 = 4$ geometrically distinct dual brake orbits on Σ .

If there are exactly 5 closed characteristics on Σ . By Theorem 1.1, four closed characteristics of them must be brake orbits after suitable time translation, then the fifth, say y , must be brake orbits after suitable time translation, otherwise $Ny(\cdot)$ will be the sixth geometrically distinct closed characteristic on Σ which yields a contradiction. Hence all closed characteristics on Σ must be brake orbits on Σ . By the same argument we can prove that all closed characteristics on Σ must be dual brake orbits on Σ . Then by the argument in the second paragraph of the proof of this theorem, all these five closed characteristics on Σ must be symmetric. Hence all of them must be symmetric brake orbits after suitable time translation. Thus we have proved the case $n = 5$ of Theorem 1.2 and the proof of Theorem 1.2 is complete. ■

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